Complementing Semi-Deterministic Buchi Automata

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Logic Automata Games

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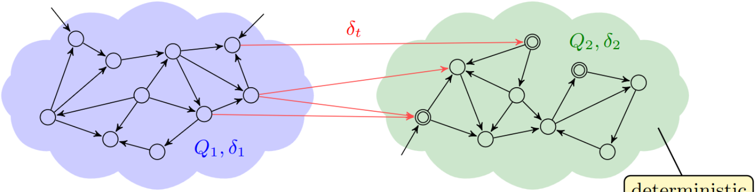
Definitions

- → A Buchi automaton $A = (Q, \Sigma, \delta, I, F)$ is called **complete** if, for each state $q \in Q$ and for each letter $a \in \Sigma$, there exists at least one successor, i.e. $|\delta(q, a)| \geq 1$.
- → A Buchi automaton A is **unambiguous** if, for each $w \in L(A)$, there exists only one accepting run over w.
- → A automaton is called **deterministic** if it has only one initial state, i.e. if $\|I\| = 1$, and if, for each reachable state $q \in Q$ and for each letter $a \in \Sigma$, there exists at most one successor, i.e. $|\delta(q, a)|1$.

Semi Deterministic Buchi Automaton

- → A Buchi automaton is semi-deterministic if it behaves deterministically from the first visit of an accepting state onward.
- → Formally, a Buchi automaton $A = (Q, \Sigma, \delta, I, F)$ is a semi-deterministic Buchi automaton if, for each $qf \in F$, the automaton $(Q, \Sigma, \delta, qf, F)$ is deterministic.


δ_t
Q_2, δ_2
Q_1, δ_1
deterministic

- $A = (Q, \Sigma, \delta, I, F)$
- $Q = Q_1 \cup Q_2$, where $F \subseteq Q_2$
- Q_1 is the non deterministic part and Q_2 is the set of states reachable by the final states including the final states i.e. the deterministic part of the SDBM.
- $\delta = \delta_1 \cup \delta_t \cup \delta_2$
  $\delta_1 : Q_1 \times \Sigma \rightarrow 2^{Q_1}$
  $\delta_t : Q_1 \times \Sigma \rightarrow 2^{Q_2}$
  $\delta_2 : Q_2 \times \Sigma \rightarrow 2^{Q_2}$,
- Relation δ_2 is deterministic: for each $q \in Q_2$ and each $a \in \Sigma, |\delta_2(q, a)| \leq 1$.
- The elements of δ_t are called transit edges. T

Runs in a SDBA

Each run ρ in the SDBA will be one of the four :

- $\rightarrow$ ρ will block (not defined further).
- $\rightarrow$ ρ will get stuck in non-deterministic part.
- $\rightarrow$ ρ will go to the deterministic part and will never visit the final state or will stop visiting the final state at some point. We say ρ is safe after it has accepted last final state or since the moment it enters Q2 if it does not visit any accepting state at all.
- $\rightarrow$ ρ is accepting run i.e. it visits the final states infinitely many times.

After reading the prefix of a word, the states visited can be divided into three parts :

- → The set $N \subseteq Q_1$ represents the runs that kept out of the deterministic part.
- → The set $C \subseteq Q_2$ represents the runs that have entered the deterministic part and that are not safe.
- → The set $S \subseteq Q_2 \backslash F$ represents the safe runs.

Intuition for compliment construction

- $\rightarrow$ Every accepting run of A stays in C after leaving N
- $\rightarrow$ If $w \notin L(A)$, every infinite run either stays in N or eventually leaves C to S and thus does not stay in C forever.

Classification of runs

- → As we keep reading a word, the run moves from N to C to S.
- → However we don't know when a run becomes safe. This guessing is done non deterministically only after leaving a final state or coming directly from a transit edge.
- → No run should stay in C forever. If a run stayed in C forever that means it was never supposed to be in compliment. That's why we introduce $B \subseteq C$ which will mimic C's behaviour and make sure that it becomes empty which we call breakpoint. (i.e. the run has become safe)

What happens in a run is moved to S too early?

Once a run is moved to S (i.e.) marked as safe, it can't visit any other final state (otherwise it wasn't safe). So in order to preserve the correctness, any transitions to final state after run is in S are made illegal.

Formal NCSB Construction

→ $-P \subseteq 2^{Q_1} \times 2^{Q_2} \times 2^{Q_2 \setminus F} \times 2^{Q_2}$

→ $-I_{\mathcal{C}} = \{(Q_1 \cap I, C, S, C) \mid S \cup C = I \cap Q_2, S \cap C = \emptyset\}$.

→ $F_{\mathcal{C}} = \{(N, C, S, B) \in P \mid B = \emptyset\}$.

→ $\delta' : P \times \Sigma \to 2^P$; $(N', C', S', B') \in \delta'((N, C, S, B), a)$

- $N' = \delta_1(N, a), C' \cup S' = \delta_t(N, a) \cup \delta_2(C \cup S, a)$
- $C' \cap S' = \emptyset$
- $S' \supseteq \delta_2(S, a)$
- $C' \supseteq \delta_2(C \backslash F, a)$
- $\forall \mathrm{q} \in C \backslash F, \delta_2(q, a) \neq \emptyset$
- if $B = \emptyset$ then $B' = C'$, and else $B' = \delta_2(B, a)\ S'$

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- for $q_3 \in Q_2 - F$, we have 4 mutually exclusive options: q_3 is only in S or only in C or both in C and B, or not present in p at all.

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- Hence, $|P| \leq 2^{|Q_1|}.3^{|F|}.4^{|Q_2 - F|}$
- For a DBA, $N \cup C \cup S$ contains exactly one state q of Q. N is empty and thus B coincides with C. If $q \in F$ then it is in both B and C, if $q \in Q_2 - F$, then it is either only is S or both B and C, thus size $\leq 2|Q_2| - |F|$

Level Rankings in Complementation

Definition

For an NBA $\mathcal{A} = (\Sigma, Q, I, \delta, F)$ and a word w, a **run graph** of $\mathcal{A}$ on w is a DAG $G_w = (V, E)$ such that

- $V \subseteq Q \times \mathbb{N}$, $(q, i) \in V$ iff there is a run $\rho = q_0 q_1 q_2 \ldots$ over A on w with $q_i = q$
- $E \subseteq V \times V$ such that $((q, i), (q', i')) \in E$ iff $i' = i + 1$ and there is a run $\rho = q_0 q_1 q_2 \ldots$ of $\mathcal{A}$ over w with $q_i = q$ and $q_{i+1} = q'$

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A vertex $(q, i) \in V$ of a graph G_w is **finite** if the set of vertices reachable from (q, i) in G_w is finite.

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3. $i = i + 2$

Ranks and Complementation of SBDAs

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$\implies$ ρ is an accepting run $\implies$ $\mathcal{A}$ accepts w, a contradiction.

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$\iff$ there is an infinite path in ${G_w}^0$ which has infinitely many vertices from $F \times \mathbb{N}$ and all vertices in this path are not safe and not finite

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$\iff$ there is an infinite path in ${G_w}^0$ which has infinitely many vertices from $F \times \mathbb{N}$ and all vertices in this path are not safe and not finite
$\implies$ All vertices in this path are there in ${G_w}^2$ and are not safe in ${G_w}^2$

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$\implies$ ${G_w}^3 \neq \emptyset$, a contradiction **QED**
Now consider the NCSB construction from a level ranking perspective. We start with an intuition for the rational run $\rho = (N_0, C_0, S_0, B_0)(N_1, C_1, S_1, B_1)(N_2, C_2, S_2, B_2) \ldots$ of $\mathcal{C}$ over a word w rejected by $\mathcal{A}$, where $(V, E) = G_w$

Rational Run

Let $\mathcal{A}$ be an SDBA, $\mathcal{C}$ be the automaton constructed by the NCSB complementation of $\mathcal{A}$, $w \notin L(A)$, and $(V, E) = G_w$ be the run graph of $\mathcal{A}$ on w. Then the run $\rho = (N_0, C_0, S_0, B_0) \ldots (N_n, C_n, S_n, B_n) \ldots$ defined as follows is a unique accepting run.

- $N_i = \{q | (q, i) \in V \text{ s.t. } q \in Q_1\}$,
- $C_i = \{q | (q, i) \in V \text{ s.t. } q \in Q_2 \text{ and the rank of } (q, i) \text{ is } 2\}$,
- $S_i = \{q | (q, i) \in V \text{ s.t. } q \in Q_2 \text{ and the rank of } (q, i) \text{ is } 1\}$,
- $B_i \subseteq C_i$

Correctness

We now establish that the automaton $\mathcal{C}$ is an unambiguous automaton that recognizes the complement language of $\mathcal{A}$ by showing

1. $\mathcal{C}$ does not accept a word that is accepted by $\mathcal{A}$,
2. for words that are not accepted by $\mathcal{A}$, the run inferred in previous slide is accepting
3. for words that are not accepted by $\mathcal{A}$, this is the only accepting run of $\mathcal{C}$ over w

Accepting words in $\mathcal{A}$

Lemma:
Let $\mathcal{A}$ be an SDBA, $\mathcal{C}$ be constructed by the NCSB complementation of $\mathcal{A}$, and $w \in \mathcal{L}(\mathcal{A})$ be a word in the language of $\mathcal{A}$. Then $\mathcal{C}$ does not accept w.

Proof. Assume for contradiction that $\mathcal{C}$ accepts w. Let $\rho' = (N_0, C_0, S_0, B_0) \ldots (N_n, C_n, S_n, B_n) \ldots$ be an accepting run of $\mathcal{C}$ over w
$\rho = q_0 q_1 \ldots$ be an accepting run of $\mathcal{A}$ over $w \implies \exists i$ s.t $q_i \in F \implies q_i \in C_i$. Hence $\forall j \geq i$, $q_j \in C_j \cup S_j$. We look at the following case distinction.

- $\{$**Case 1:** $\forall j \geq i,\ q_j \in C_j\}$
- $\{$**Case 2:** There is a $j \geq i$ such that $q_j \in S_j\}$

- $\{$**Case 1:** $\forall j \geq i,\ q_j \in C_j\}$
  As ρ' is accepting, there is a breakpoint ($B_j = \varnothing$) for some $j \geq i$. For such a j we have that $q_{j+1} \in B_{j+1} = C_{j+1}$ and, moreover, that $q_k \in B_k$ for all $k \geq j+1$. Thus, $B_k \neq \varnothing$ for all $k \geq j+1$ and ρ' visits only finitely many accepting states (contradiction).
- $\{$**Case 2:** There is a $j \geq i$ such that $q_j \in S_j\}$
  But then $q_k \in S_k$ holds for all $k \geq j$ by construction. However, as ρ is accepting, there is an $l \geq j$ such that $q_l \in F$, which contradicts $q_l \in S_l$ (contradiction).

Accepting run for w $\notin \mathcal{L}(\mathcal{A})$

Lemma:

Let $\mathcal{A}$ be an SDBA, $\mathcal{C}$ be the automaton constructed by the NCSB complementation of $\mathcal{A}$, $w \notin L(A)$, and $(V, E) = G_w$ be the run graph of $\mathcal{A}$ on w. Then the following run ρ is accepting
$\rho = (N_0, C_0, S_0, B_0)(N_1, C_1, S_1, B_1)\ldots(N_n, C_n, S_n, B_n)\ldots$

- $N_i = \{q|(q, i) \in V \text{ s.t. } q \in Q_1\}$,
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Clearly this is a valid run as it follows the transition rules in $\mathcal{C}$. The value of B is fully determined by the C and previous B. Hence the above run is a unique valid run, we just need to show that it is accepting.

Proof. Let us assume for contradiction, that there are only finitely many breakpoints reached, i.e. $\exists i \in \mathbb{N}$ such that $\forall j \geq i$,
$\varnothing \neq B_i \subseteq C_i$
$\implies \bigcup_{j \geq i} B_j \times \{j\}$ is infinite
Since $B \neq \varnothing$ at all times, no new runs are copied into B
Hence,$\bigcup_{j \geq i} B_j \times \{j\} = B_i \times \{i\} \cup \bigcup_{j \geq i} (\delta(B_j, w[j]) - S_{j+1}) \times \{j+1\}$
$=$\{reachable vertices from $B_i \times \{i\}$ in G_w which are not safe\}
$=$\{reachable vertices from $B_i \times \{i\}$ in G_w^1\}
$\implies$ there are infinite vertices reachable from $(B_i \times \{i\})$ in ${G_w}^1$

Since ${G_w}^1$ is a finitely branching tree , From Koning's lemma, $\exists$ an infinite path in ${G_w}^1$ from one of the vertices $(b, i) \in (B_i \times \{i\})$

But $b \in C_i$, This contradicts the assumption that the rank of these vertices is 2, i.e., they are finite in ${G_w}^1$.

Lemma:

Let $\mathcal{A}$ be an SDBA, $\mathcal{C}$ be the automaton constructed by the NCSB complementation of $\mathcal{A}$, $w \notin L(A)$, and $(V, E) = G_w$ be the run graph of $\mathcal{A}$ on w. Then the run
$\rho = (N_0, C_0, S_0, B_0) \ldots (N_n, C_n, S_n, B_n) \ldots$
is not accepting if it doesn't satisfy

- $N_i = \{q | (q, i) \in V \text{ s.t. } q \in Q_1\}$,
- $C_i = \{q | (q, i) \in V \text{ s.t. } q \in Q_2 \text{ and the rank of } (q, i) \text{ is } 2\}$,
- $S_i = \{q | (q, i) \in V \text{ s.t. } q \in Q_2 \text{ and the rank of } (q, i) \text{ is } 1\}$,

As the N part always tracks the reachable states in Q1 and the $C \cup S$ part always tracks the reachable states in Q2 correctly by construction. We only need to consider two possible cases.

- $\exists q \in C_i$ where rank of (q, i) is 1, i.e. its a safe vertex
- $\exists q \in S_i$ where rank of (q, i) is 2, i.e. its a non-safe vertex

By construction, since $q_i = q \in Q_2$ exists in G_w, $\exists$ unique maximal path $(q_i, i)(q_{i+1}, i+1)(q_{i+2}, i+2)\ldots$,

- **Case 1** $\exists$ safe vertex $(q, i) \in V$ such that $q \in C_i$

  Since (q, i) is safe, $\nexists$ any accepting state in the path. By an inductive argument, $\forall$ vertices (q_j, j) on this path, $q_j \in C_j$.
  - If the path is finite. Let (q_k, k) be the last vertex. i.e., $\delta(q_k, w[k]) = \varnothing$ and $q_k \in C$. Then by construction rules, ρ blocks this run. Contradicting run ρ being infinite.
  - If the path is infinite, $q_k \in B_k$ for some $k \geq i$. Then $q_j \in B_j$ for all $j > k$ with (q_j, j) on this path. Therefore, ρ cannot be accepting.
- **Case 2** $\exists$ non-safe vertex $(q, i) \in V$ such that $q \in S_i$

  Since (q, i) is non-safe, $\exists$ an accepting state q_k in the path. By an inductive argument, $\forall$ vertices (q_j, j) on this path, $q_j \in S_j$. But this implies $q_k \in S_k$ (contradiction).

Theorem 1

Let $\mathcal{C}$ be an SDBA and $\mathcal{C}$ be the automaton constructed by the NCSB complementation of $\mathcal{A}$. Then $\mathcal{C}$ is an unambiguous B¨uchi automaton that recognises the complement of the language of $\mathcal{A}$.