Stability Analysis of SIR Model

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1. Introduction

In this report, I will be analytically analysing an SIR epidemic model with a non-linear incidence rate. I will be analysing its fixed points, the stability of the fixed points, limit cycles and drawing conclusions from it that are significant to real life.

2. Introduction to SIR Models

2.1. What is an SIR Model

SIR stands for Susceptible-Infected-Recovered model. It is an epidemiological model which is used to calculate the number of people who are susceptible to the disease , infected by the disease and have recovered from the disease.

2.2. Basic SIR Model

The most basic SIR model is the Kermack-McKendrick Model. It was introduced in 1927 to explain the rise and fall of infected patients in plagues. This model assumes fixed population i.e. no new borns, no deaths ; completely homogeneous population and instantaneous incubation period of infected individuals.

$$\begin{aligned} S' &= -\beta SI \\ I' &= \beta SI - \gamma I \\ R' &= \gamma I \end{aligned}$$

- → $S(t)$ defines the number of susceptible individuals at time t. β is the rate of infection. βSI is the number of people getting infected and hence getting removed from the susceptible population.
- → $I(t)$ defines the number of Infected individuals at time t. It is equal to the number of people getting infected i.e. βSI minus the number of people who are getting recovered i.e. γI. γ is the recovery rate.
- → $R(t)$ defines the number of Recovered individuals at time t. It is equal to the recovery rate multiplied by the number of infected people.

2.3. A Non Linear SIR Model

While the previous model is easy to understand, it is not the most accurate because of the number of assumptions it makes. In our report we will use a more advanced SIR model. This model is non-linear.

$$\begin{aligned} \frac{dS}{dt} &= B - dS - \frac{kI^l S}{1+\alpha I^h} + vR \\ \frac{dI}{dt} &= \frac{kI^l S}{1+\alpha I^h} - (d+\gamma)I \\ \frac{dR}{dt} &= \gamma I - (d+v)R \end{aligned}$$

2.3.1 Incidence rate

Incidence rate is the measure of how frequently new cases of a disease occur over a specified period of time. In the previous model, the incidence rate was βSI. In this model, we have a non-linear incidence rate $\frac{kI^lS}{1+\alpha I^h}$. Here kI^l is a measure of how infectious the disease is. $1/\left(1+\alpha I^h\right)$ accounts for behavioural changes when a disease in spreading. For example, Isolation of infected individuals and non infected individuals.

2.3.2 Parameters

→ B is rate at which new individuals are getting added to the population (Recruitment Rate)
→ ν is the rate at which individuals lose immunity and become susceptible again.
→ γ is the recovery rate of individuals.
→ l and h are positive constants and α is non-negative
This model is much more accurate than the previous one since this takes in account the inhibitions of suspected individuals, death rate, birth rate etc.

3. SIR Model used in this paper

In this paper we will be using a simplified version of the non-linear model discussed above with $l = 2$

$$\frac{dS}{dt} = B - dS - \frac{kI^2S}{1+\alpha I^h} + vR \tag{1}$$
$$\frac{dI}{dt} = \frac{kI^2S}{1+\alpha I^h} - (d+\gamma)I \tag{2}$$
$$\frac{dR}{dt} = \gamma I - (d+v)R \tag{3}$$

Adding 1, 2 and 3

$$\begin{aligned} S' + I' + R' = N' &= B - dS - \frac{kI^lS}{1+\alpha I^h} + vR + \frac{kI^lS}{1+\alpha I^h} - (d+\gamma)I + \gamma I - (d+v)R \\ &= B - dN \end{aligned}$$

As $t \to \infty$, total population tends to a constant i.e. $N' = 0$ and $\exists N_0$ such that $B = dN_0$

$$S + I + R = N_0$$
$$S = N_0 - I - R$$

Replacing S with $N_0 - I - R$

$$\frac{dI}{dt} = \frac{kI^2}{1+\alpha I^2}(N_0 - I - R) - (d+\gamma)I \tag{4}$$
$$\frac{dR}{dt} = \gamma I - (d+v)R \tag{5}$$

Dividing both equations by $d + \nu$, we get

$$\frac{dI}{dt} = \frac{I^2}{1+pI^2}(A - I - R) - mI \tag{6}$$
$$\frac{dR}{dt} = qI - R \tag{7}$$

Where

$$p = \frac{\alpha(d+v)}{k}, \quad A = N_0\sqrt{\frac{k}{d+v}}$$
$$m = \frac{d+\gamma}{d+v}, \quad q = \frac{\gamma}{d+v}$$

4. Fixed Points

4.1. Trivial Fixed point

(0,0) is a trivial fixed point.

4.2. Non-Trivial Fixed point

$$qI - R = 0 \tag{8}$$
$$\frac{I}{1+pI^2}(A - I - R) - m = 0 \tag{9}$$

Putting $R = qI$ in (9)

$$\frac{I}{1+pI^2}(A - I - qI) - m = 0$$
$$AI - I^2 - qI^2 - m1 - mpI^2 = 0$$
$$I^2(-1 - q - mp) + AI - m = 0$$
$$I^2(1 + q + mp) - AI + m = 0$$

$$\Delta = A^2 - 4m(mp + q + 1)$$

This system has two non trivial fixed points when $A^2 > 4m(mp + q + 1)$
In that case the two fixed points are,

$$I_1 = \frac{A - \sqrt{A^2 - 4m(mp+q+1)}}{2(mp+q+1)}, \quad R_1 = qI_1 \tag{10}$$
$$I_2 = \frac{A + \sqrt{A^2 - 4m(mp+q+1)}}{2(mp+q+1)}, \quad R_2 = qI_2 \tag{11}$$

5. Stability Analysis

$$Jacobian = \begin{bmatrix} \frac{x\left(A - ApI_2^2 - 2x - qx + qI_2^3p\right)}{(1+px^2)^2} & \frac{-x^2}{1+px^2} \\ q & -1 \end{bmatrix} \tag{12}$$

5.1. Stability Analysis of trivial fixed point

$$J_{(0,0)} = \begin{bmatrix} 0 & 0 \\ q & -1 \end{bmatrix}$$
$$\Delta = 0$$
$$\tau = -1$$

It is a **Sink** and a global attractor.

5.2. Stability Analysis of I_1

$$J_{(I_1,R_1)} = \begin{bmatrix} \frac{I_1\left(A - ApI_1^2 - 2I_1 - qI_1 + qI_1^3p\right)}{\left(1+pI_1^2\right)^2} & -I_1^2/\left(1+pI_1^2\right) \\ q & -1 \end{bmatrix}$$
$$\Delta = -\frac{I_1\left(A - ApI_1^2 - 2I_1 - qI_1 + qI_1^3p\right)}{\left(1+pI_1^2\right)^2} + \frac{qI_1^2}{1+pI_1^2}$$

All the parameters are always positive. Refer to Section 2.3.2 and eqns 6 and 7 where we defined the re-scaled parameters.
The first term is of a larger magnitude than the second term since I_1 is negative and the second term is contained in the first term. This means that the negative term dominates, making the determinant negative.

Since $\Delta < 0$, it is a **Saddle Point**

5.3. Stability Analysis of I_2

$$\Delta = -\frac{I_2\left(A - ApI_2^2 - 2I_2 - qI_2 + qI_2^3p\right)}{\left(1 + pI_2^2\right)^2} + \frac{qI_2^2}{1 + pI_2^2}$$

Since I_2 is positive, determinant is positive. It can be a spiral, center, node or star. But its not the type of fixed we are concerned but the stability of the fixed point. To do that, we will look at trace.

$$\tau = \left(qp - p^2\right) I_2^4 - ApI_2^3 - (2 + q + 2p)I_2^2 + AI_2 - 1 \tag{13}$$

For I_2 to be stable, we need the τ to be negative. This will be possible in few cases.

5.3.1 Case 1 : $m \leq 1$

From eqn 11, is m is less is magnitude, I_2 increases in magnitude which in turn makes τ negative (by boosting the I_2^3 term.)

$$\begin{aligned} m = \frac{d+\gamma}{d+\nu} &\leq 1 \\ \implies d + \gamma &\leq d + \nu \\ \implies \gamma &< \nu \end{aligned}$$

CONCLUSION : (I_2, R_2) is stable when losing immunity rate is stronger than recovery rate

5.3.2 Case 2 : $q < (2mp + 1)/(m - 1)$

If we put back the original parameters in this equation, we get :

$$\left(\gamma^2 - \gamma v - d^2 - 2dv - v^2\right) k - 2\alpha(d + v)^2(d + \gamma) < 0$$

For this to be true, k should be very less.

CONCLUSION : (I_2, R_2) is stable when contact rate k is really small

6. Limit Cycles

Now we look at any possible limit cycles in this system.

6.1. Condition for no Limit cycles

Dulac's Criterion : If there exists a function such that its in-variance is of the same sign in the whole set, then there exist no limit cycles.

Let that function be $D(I) = \left(1 + pI^2\right)/I^2$

Invariance : $\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} = -(1 + p + mp) + \frac{m-1}{I^2}$

From the invariance equation, its evident that if $m \leq 1$, then the second term will be negative (or 0) which will make the in-variance always negative.

Hence if $m \leq 1$, limit cycles dont exist.

After some manipulation we can also see $I_2 A(mp + 1 - q) > 2m(mp + 1)$ as an condition for non-existence of limit cycles.

6.1.1 Conclusion

If losing immunity rate is stronger than recovery rate OR q is small and A is large which means either population size and contact rate (k) is large and recovery size (γ) is small ; then limit cycles do not exist.

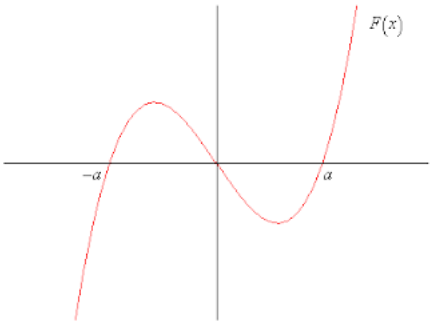
6.2. Condition for limit cycles

To find conditions for Limit cycles, we will introduce Lienard Theoram.

Lienard's Theoram : The system $\dot{x} = y - F(x)$; $\dot{y} = -g(x)$ will have atmost one limit cycle if the following conditions are fulfilled.

1. $F(x) = \int_0^x f(u)du$
2. $g(x) > 0$ for all $x > 0$
3. $\lim_{x\to\infty} F(x) := \lim_{x\to\infty} \int_0^x f(\xi)d\xi = \infty$;
4. $F(x)$ has exactly one positive root at some value p, where $F(x) < 0$ for $0 < x < p$ and $F(x) > 0$ and monotonic for $x > p$.

Basically $f(x)$ should be of form



6.3. Turning our system into a Lienard system

$$\frac{dx}{d\tau} = y - F(x)$$
$$\frac{dy}{d\tau} = -g(x)$$

where

$$X = I \quad Y = g_0(I) - g_1(I)R$$
$$d\tau = g_1(X)dt$$
$$g_1(I) = \frac{I^2}{1+pI^2}. \quad g_0(I) = \frac{I^2(A-I)}{1+pI^2} - mI,$$
$$F(x) = (mp+p+1)x + \frac{(1-m)x}{I_2\,(I_2+x)},$$
$$y = v - (mp+p+1)I_2 + \frac{1-m}{I_2}.$$
$$g(x) = \frac{(mp+q+1)x\,(x+I_2-I_1)}{(x+I_2)\,g_1\,(x+I_2)}.$$
$$x = X - I_2$$

Now to make the system a true Lienard system, conditions on f(x) and g(x) need to be satisfied. (Refer to 6.2)

Suppose the conditions for no limit cycles are not satisfied i.e. $m \geq 1$ and $I_2A(mp+1-q) < 2m(mp+1)$; then there is atmost one limit cycle if $h(x) \leq 0$ for $x \in \left[I_1, \sqrt{(m-1)/(mp+p+1)}\right]$
where $h(x) = -p(mp+p+1)(1+mp+q)x^6 + \left(-p^2m - 2pq + 6mp + 5m^2p^2 - 2p + 1 + 4mpq + q\right)x^4 - 2A(2mp+1)x^3 + \left(mq - q + mp - 1 + 5m^2p + 4m\right)x^2 - m(m-1)$

6.3.1 Conclusion

Condition for Lienard cycle is satisfied when A and m are large. This means large population and good recovery rate implies the existence of atmost one limit cycle.

7. Final Conclusion

- → (0,0) is the global attractor
- → Under certain conditions, two other fixed points can arise. One will be a saddle point. Which means that whether the disease dies out or blows up depends on the initial value.
- → The other fixed point can be stable or unstable. A stable limit point represents that even though the disease is persisting, it isn't growing or decreasing i.e endemic.
- → A limit cycle can also arise under certain conditions (large population and good recovery rate) which means that the disease is coming back in periodic manner.

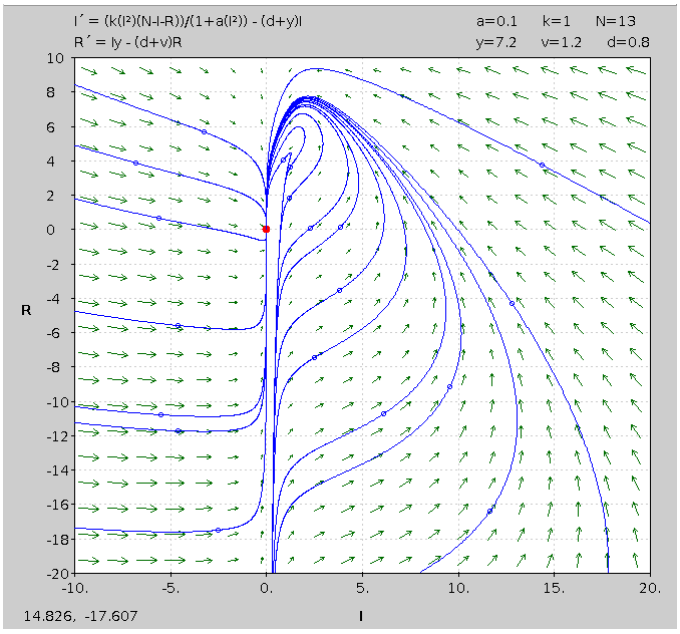


Figure 1: Only equilibrium is (0,0) which is a nodal sink. No endemic equilibrium.

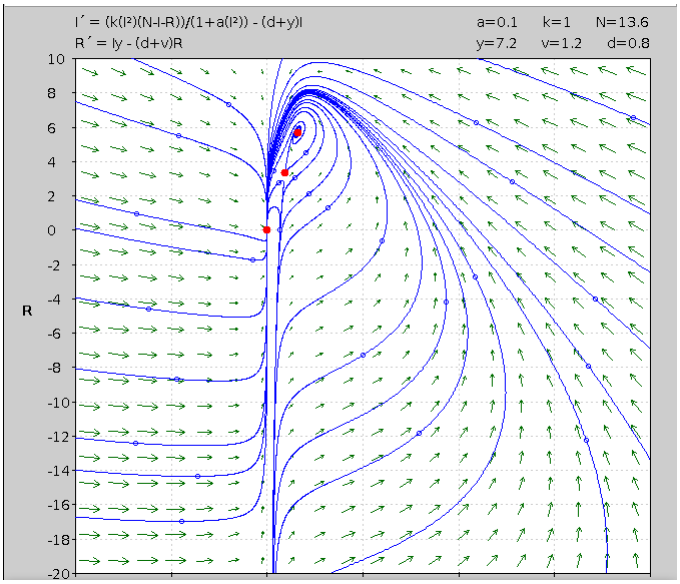


Figure 2: Spiral source, Saddle point and Nodal sink. Since its a spiral source, the endemic is unstable and for most initial values the disease will die out.

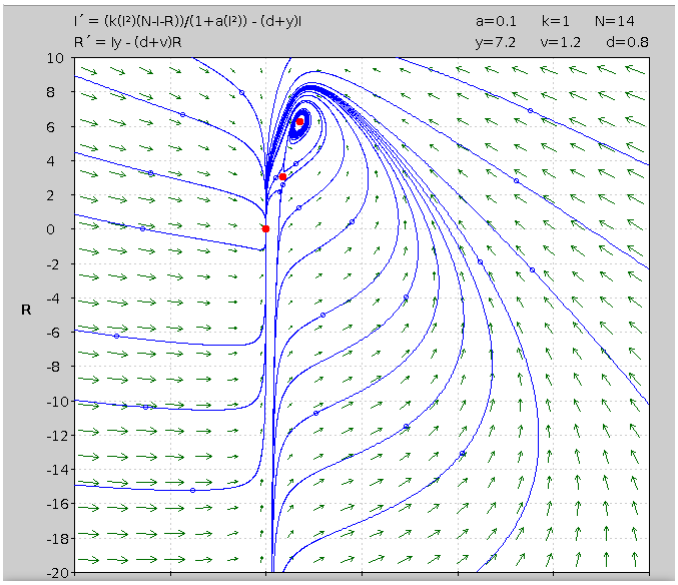


Figure 3: Nodal sink (0,0) ; Saddle and a Spiral Sink. This means that disease can persist. Hence endemic.

8. Primary Resources

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